THE SUMMATION OF TWO SERIES.

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As the following series occur in the solution of problem 121, Calculus, the method of summing them, though neither difficult nor tedious, may be of interest to many of our readers.

$$y=1-\frac{n^2x^2}{2\,!}+\frac{n^2(n^2+8)x^4}{4!}-\frac{n^2\,(n^4+40n^2+184)x^6}{6!}+\ldots.(A).$$

$$y=\frac{nx}{1!}-\frac{n(n^2+2)x^3}{3\,!}+\frac{n(n^4+20n^2+24)x^5}{5\,!}$$

$$-\frac{n(n^6+70n^4+784n^2+720)x^7}{7\,!}+.\ \ ..(B).$$

The general term of both being

$$a_{r+4}=-\frac{r(r+1)_r+[2(r+2)^2+n^2]a_{r+2}}{(r+3)(r+4)}.$$

From (A), $dy/dx=-n^2x+\dfrac{n^2(n^2+8)x^3}{3\,!}-\dfrac{n^2(n^4+40n^2+184)\,x^5}{5\,!}+\ldots.(1).$

$$d^2y/dx^2=-n^2+\frac{n^2(n^2+8)x^2}{2\,!}-\frac{n^2(n^4+40n^2+184)\,x^4}{4\,!}+\ldots.(2).$$

n^2 times (A) gives $n^2y=n^2-\dfrac{n^4x^2}{2\,!}+\dfrac{n^4(n^2+8)x^4}{4\,!}-\ldots.(3).$

$2x(1+x^2)$ times (1) gives $2x(1+x^2)(dy/dx)=-2n^2x^2+\dfrac{2n^2(n^2+2)\,x^4}{3\,!}-\ldots.(4).$

$(1+x^2)^2$ times (2) gives

$$(1+x^2)^2(d^2y/dx^2)=-n^2+\frac{n^2(n^2+4)x^2}{2\,!}-\frac{n^2(n^4+16n^2+16)x^2}{4\,!}+\ldots.(5).$$

(3)+(4)+(5) gives $(1+x^2)^2(d^2y/dx^2)+2x(1+x^2)(dy/dx)+n^2y=0\ldots$ (6).

From (B), $dy/dx=$

$$n-\frac{n(n^2+2)x^2}{2\,!}+\frac{n(n^4+20n^2+24)x^4}{4\,!}-\frac{n(n^6+70n^4+784n^2+720)x^6}{6\,!}+\ldots.(7).$$

$$d^2y/dx^2=-\frac{n(n^2+2)x}{1\,!}+\frac{n(n^4+20n^2+24)x^3}{3\,!}$$

$$-\frac{n(n^6+70n^4+784n^2+720)x^5}{5\,!}\ldots.(8).$$

n^2 times (B) gives

$$n^2y=\frac{n^3x}{1\,!}-\frac{n^3(n^2+2)x^3}{3\,!}+\frac{n^3(n^4+20n^2+24)x^5}{5\,!}-\ldots.(9).$$

$2x(1+x^2)$ times (7) gives

$$2x(1+x^2)(dy/dx)=2nx-\frac{2n^3x^3}{2\,!}+\frac{3n^3(n^2+8)x^5}{4\,!}-\ldots.(10).$$

$(1+x^2)^2(d^2y/dx^2)=$

$$-\frac{n(n^2+2)x}{1\,!}+\frac{n^3(n^2+8)x^3}{3\,!}-\frac{n^3(n^4+30n^2+104)x^5}{5\,!}+\ldots.(11).$$

$(9)+(10)+(11)$ gives $(1+x^2)^2(d^2y/dx^2)+2x(1+x^2)(dy/dx)+n^2y=0$,

the same as $(6)\ldots.(12)$.

Multiplying (6) and (12) through by $2(dy/dx)$, and integrating, we get $(1+x^2)^2(dy/dx)^2+n^2y^2+C=0$.

When $x=0$, from (A), $y=1$, $dy/dx=0$.

When $x=0$, from (B), $y=0$, $dy/dx=n$.

$\therefore$ In either case, $C=-n^2$. $\therefore$ $(1+x^2)^2(dy/dx)^2=n^2(1-y^2)$.

$$(1+x^2)dy/dx=\pm n\sqrt{(1-y^2)} \text{ or } \frac{dy}{\sqrt{(1-y^2)}}=\pm\frac{ndx}{1+x^2}.$$

$\therefore$ $\sin^{-1}y=n\tan^{-1}x+D$, or $\sin^{-1}y=n\cot^{-1}x+D$.

From (A), when $x=0$, $y=1$.

$\therefore$ $D=\frac{1}{2}\pi$. $\therefore$ $\sin^{-1}y=n\tan^{-1}x+\frac{1}{2}\pi=n\cot^{-1}x+\frac{1}{2}\pi$.

$\therefore$ $y=\sin(n\tan^{-1}x+\frac{1}{2}\pi)=\sin(n\cot^{-1}x+\frac{1}{2}\pi)$.

$\therefore$ $y=\cos(n\tan^{-1}x)=\cos(n\cot^{-1}x)$.

From (B), when $x=0$, $y=0$.

$\therefore$ $D=0$. $\therefore$ $\sin^{-1}y=n\tan^{-1}x=n\cot^{-1}x$.

$\therefore$ $y=\sin(n\tan^{-1}x)=\sin(n\cot^{-1}x)$.